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# Chiral Topologically Massive Gravity and Extremal B-F Scalars

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## Abstract

At a critical "chiral" coupling, topologically massive gravity with a negative cosmological constant exhibits several unusual features, including the emergence of a new logarithmic branch of solutions and a linearization instability for certain boundary conditions. I show that at this coupling, the linearized theory may be parametrized by a free scalar field at the Breitenlohner-Freedman bound, and use this description to investigate these features.

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# 1 Some puzzles

Topologically massive gravity [1]—three-dimensional Einstein gravity supplemented by a gravitational Chern-Simons term in the action—provides an interesting playground in which to explore quantum gravity. In contrast to pure Einstein gravity in three dimensions, topologically massive gravity contains a propagating physical degree of freedom. It nevertheless appears to be renormalizable [2–4], perhaps offering a rare instance of a theory of spacetime geometry that can be treated by methods of conventional quantum field theory.

With the addition of a negative cosmological constant, topologically massive gravity also provides a new realm in which to investigate the AdS/CFT correspondence [5–7]. In this setting, though, the theory has a dangerous instability: the local degree of freedom contributes to the energy with a sign opposite to that of the BTZ black hole [8], suggesting the absence of a ground state. Using the supersymmetric extension of topologically massive gravity, one can prove a positive energy theorem [9, 10], but only in the sector containing no black holes.

As Li, Song, and Strominger have noted, however, the structure of the theory changes dramatically at a special "chiral" value of the coupling constants [7]. At this coupling, the propagating "massive graviton" modes become pure gauge, and can be eliminated by diffeomorphisms [11]. This presents a puzzle, since the constraint analysis shows the continued existence of a local degree of freedom [12, 13]. The answer is partially understood [14, 15]: precisely at the chiral coupling, a new set of local "logarithmic" modes appears in the linearized theory.* These modes violate the Brown-Henneaux boundary conditions [17] usually imposed in (2+1)-dimensional asymptotically AdS gravity, and it has been argued that they can be eliminated by imposing suitable boundary conditions. If so, topologically massive gravity at the chiral coupling would become a truly chiral theory, closely related to a chiral half of ordinary Einstein gravity.

But this leads to another puzzle. As demonstrated in [18], the logarithmic modes at the chiral coupling can be generated, at least in the linearized theory, from strictly local initial data. The imposition of Brown-Henneaux boundary conditions then becomes a teleological choice, a restriction on initial data "now" on the basis of its behavior in the future. A possible resolution, proposed in [19], could come from a linearization instability: although linearized initial data may have compact support, there is evidence that at the next order in perturbation theory, such data may violate suitably strong boundary conditions.

Any resolution of these puzzles requires an understanding of the interplay between boundary conditions and nonlinearity in topologically massive gravity. In this paper, I find the general solution to the linearized field equations in a Poincaré coordinate patch, and show that it can be parametrized by solutions—or initial data—of a free scalar field whose mass lies at the Breitenlohner-Freedman bound [20]. I then use this parametrization to investigate the questions of boundary behavior and linearization stability.

# 2 Linearized topologically massive gravity and B-F scalars

Consider a free scalar field $\varphi$ with mass $m$ in three-dimensional anti-de Sitter space. In a Poincaré coordinate patch† (with units $\ell = 1$), the metric is

$$d\bar{s}^2 = \frac{1}{z^2}\left(2dx^+dx^- + dz^2\right) \tag{2.1}$$

*The appearance of logarithmic modes in *exact* pp-wave solutions at the chiral coupling was noted earlier in [16].

†A Poincaré patch does not cover the whole of AdS, but we will be interested in local initial data, for which such a coordinate choice is sufficient and greatly simplifies computations.

with $x^{\pm} = \frac{1}{\sqrt{2}}(x \pm t)$, and the Klein-Gordon equation is simply

$$\left[2z^2\partial_+\partial_- + (N-1)^2 - (m^2+1)\right]\varphi = 0, \tag{2.2}$$

where $N = z\partial_z$. Near the conformal boundary $z = 0$, solutions behave as

$$\varphi \sim z^{1\pm\sqrt{1+m^2}}. \tag{2.3}$$

One solution thus seems to disappear at the Breitenlohner-Freedman bound $m^2 = -1$. This behavior is well-understood, though: precisely at $m^2 = -1$, a new solution appears with an asymptotic behavior

$$\varphi \sim z \ln z. \tag{2.4}$$

This is exactly the behavior of linearized topologically massive gravity noted by Grumiller and Johansson at the chiral coupling [14]. Indeed, the two theories are intimately related. Let $\varphi$ be an extremal B-F scalar, that is, one satisfying (2.2) with $m^2 = -1$, and define $X = \varphi/z^3$, so

$$\left[2z^2\partial_+\partial_- + (N+2)^2\right]X = 0. \tag{2.5}$$

Consider a linear excitation $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ of the metric around AdS space, and choose Gaussian normal coordinates in $z$,

$$h_{+z} = h_{-z} = h_{zz} = 0, \tag{2.6}$$

as in the Fefferman-Graham expansion [21]. As I show in the appendix, the perturbation

$$\begin{aligned} h_{++} &= -2\partial_+^2(N+2)X \\ h_{+-} &= 2\partial_+\partial_-(N+2)X = -\frac{1}{z^2}N(N+2)^2X \\ h_{--} &= -2\partial_-^2(N-2)X \end{aligned} \tag{2.7}$$

then exactly solves the linearized field equations of topologically massive gravity at the chiral coupling, thus parametrizing solutions by a scalar at the B-F bound.

More than that: as demonstrated in the appendix, the *general* solution in a Poincaré coordinate patch can be obtained by adding to (2.7) a solution of the form

$$\begin{aligned} \tilde{h}_{++} &= \frac{\partial_+{}^2 c}{z^2} + \frac{1}{4}\partial_+{}^3\partial_-(2a-b-c) + \gamma^+ \ln z \\ \tilde{h}_{+-} &= \frac{\partial_+\partial_- a}{z^2} + \frac{1}{4}\partial_+{}^2\partial_-{}^2(2a-b-c) \\ \tilde{h}_{--} &= \frac{\partial_-{}^2 b}{z^2} + \frac{1}{4}\partial_+\partial_-{}^3(2a-b-c), \end{aligned} \tag{2.8}$$

where $a$, $b$, and $c$ are arbitrary functions of $x^+$ and $x^-$ and $\gamma^+$ depends only on $x^+$. The linearized curvature of $\tilde{h}$ depends only on $\gamma^+$, so the remaining piece of $\tilde{h}$ is formally equivalent to a diffeomorphism; in fact, it is the most general diffeomorphism that preserves the gauge condition (2.6) at first order.

The general solution of linearized topologically massive gravity at the chiral coupling in a Poincaré coordinate patch thus consists of three pieces: a dynamical degree of freedom parametrized by an extremal B-F scalar; a formal diffeomorphism (that need not satisfy the boundary conditions to make it "pure gauge"); and a chiral logarithmic term $h_{++} = \gamma^+(x^+)\ln z$ corresponding to the pp-wave of [16, 18].

## 3 Boundary values

At an initial time, the scalar field $X$ may be chosen to have compact support away from the conformal boundary. Indeed, we can specify arbitrary initial data for $X$ and $\partial_t X$ and evolve it forward with the Klein-Gordon equation. As the system evolves, though, the field will reach the conformal boundary in a finite time, and we may ask about compatibility with boundary conditions. Note that we risk making a teleological argument here, that is, limiting initial data "now" on the basis of what the field will do in the future.

Anti-de Sitter space is not globally hyperbolic, so initial data is not enough to determine a unique evolution. For the case of a free scalar, however, Ishibashi and Wald have shown that there is a finite-dimensional family of "nice" evolution laws, where niceness includes the existence of a positive energy, compatibility with time translation symmetry, and a set of convergence conditions [22]. In particular, for a scalar field $\varphi = z^3 X$ at the Breitenlohner-Freedman bound, the asymptotic behavior near $z = 0$ is

$$\varphi \sim a_0 z + b_0 z \ln z + \ldots, \tag{3.1}$$

and the allowed evolution laws correspond uniquely to choices of the ratio $a_0/b_0$. (Positivity imposes an interesting limit on this ratio, whose implications have not, as far as I know, been explored in the context of topologically massive gravity.)

Each choice of an evolution law gives rise to a Greens function, and in principle one could use such a function to determine the future boundary behavior of $X$. Let us take a shortcut, and directly evaluate $X$—and hence $h_{\mu\nu}$—near $z = 0$, keeping in mind that we are really looking at the future evolution of our compact initial data. Since $X = \varphi/z^3$, the appropriate form of (3.1) is now

$$X \sim \frac{\alpha_0}{z^2} + \frac{\beta_0 \ln z}{z^2} + \alpha_1 + \beta_1 \ln z + \ldots, \tag{3.2}$$

where the coefficients are functions of $x^+$ and $x^-$. The equations of motion (2.5) then yield

$$\alpha_1 = \frac{1}{2}\partial_+\partial_-(\beta_0 - \alpha_0), \qquad \beta_1 = -\frac{1}{2}\partial_+\partial_-\beta_0. \tag{3.3}$$

This, in turn, implies from (2.7) that

$$\begin{aligned} h_{++} &\sim -\frac{2{\partial_+}^2\beta_0}{z^2} - {\partial_+}^3\partial_-(\beta_0 - 2\alpha_0) + 2{\partial_+}^3\partial_-\beta_0 \ln z + \ldots \\ h_{+-} &\sim \frac{2\partial_+\partial_-\beta_0}{z^2} + {\partial_+}^2{\partial_-}^2(\beta_0 - 2\alpha_0) - 2{\partial_+}^2{\partial_-}^2\beta_0 \ln z + \ldots \\ h_{--} &\sim -\frac{2{\partial_-}^2(\beta_0 - 4\alpha_0)}{z^2} + \frac{8{\partial_-}^2\beta_0 \ln z}{z^2} + \partial_+{\partial_-}^3(3\beta_0 - 2\alpha_0) - 2\partial_+{\partial_-}^3\beta_0 \ln z + \ldots \end{aligned} \tag{3.4}$$

These asymptotics clearly violate standard boundary conditions, including both the original Brown-Henneaux boundary conditions for (2+1)-dimensional gravity [17] and the weaker logarithmic boundary conditions of [13,23]. We have not yet looked at the general solution, however; that is, we have not yet added in a term of the form (2.8). By inspection, we can cancel the $1/z^2$ terms in (3.4) by choosing $c = 2\beta_0$, $a = -2\beta_0$, and $b = 2\beta_0 - 8\alpha_0$. The $\ln z/z^2$ term in $h_{--}$ cannot be canceled, though, and must be eliminated by imposing the condition

$${\partial_-}^2\beta_0 = 0. \tag{3.5}$$

It is then easy to check that

$$
\begin{aligned}
&h_{++} \sim A(x^+, x^-) + B(x^+) \ln z + \ldots \\
&h_{+-} \sim 0 + \ldots \\
&h_{--} \sim 0 + \ldots
\end{aligned}
\tag{3.6}
$$

consistent with the logarithmic asymptotics of [13, 23].

At first sight, the condition (3.5) appears to be a teleological restriction on our initial data. As noted earlier, however, a scalar field in anti-de Sitter space does not have a unique Greens function. In particular, one can choose a Greens function for which the coefficient $b_0$ in (3.1) vanishes for all initial data. If we assume "nice" evolution of $X$ is in one-to-one correspondence with "nice" evolution of $h_{ij}$—I will return to this assumption in the conclusion—then the corresponding choice eliminates $\beta_0$ in (3.4).[‡]

With the proper choice of a Greens function, compact initial data for $X$ thus evolves in a way that respects natural Brown-Henneaux or logarithmic boundary conditions. This is still a delicate issue, though, since the cancellation of the $1/z^2$ terms in (3.4) required a "diffeomorphism mode" (2.8) which did not itself have compact support. A direct Greens function computation of the evolution of the full metric perturbation would thus still be valuable.

## 4 Linearization stability

So far, we have only considered the linear approximation to topologically massive gravity. For the most part, physicists are used to situations in which a first-order solution of a set of equations can be extended to at least a perturbative solution of the full nonlinear equations. Sometimes, however, such an extension fails: a solution of a linearized set of field equations may not be the linearization of an exact solution. In such a case, the theory is said to have a linearization instability [24].

As a simple example, consider the equation $x^2 = 0$. If we expand around $\bar{x}$, the linearized equation is $\bar{x}\delta x = 0$; thus if we choose $\bar{x} = 0$, $\delta x$ is unconstrained at linear order, although only $\delta x = 0$ is a linearization of the exact solution $x = 0$. While the instances in physics are more complicated, this example illustrates the fundamental point: if one expands around too special a background, some of the linearized equations may vanish identically, and the first constraints may be quadratic.

In ordinary general relativity, a linearization instability can occur if one expands around a background metric admitting Killing vectors. Recall that in $d$ dimensions, $d$ of the Einstein field equations are constraints; that is, when smeared against a vector $\xi^\mu$, they generate diffeomorphisms $g_{\mu\nu} \rightarrow g_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$. But if $\xi_\mu$ is a Killing vector for a background metric $\bar{g}_{\mu\nu}$, this transformation is trivial at first order, and up to possible boundary terms, the corresponding generators vanish. In other words, if the background metric admits a Killing vector, certain combinations of the field equations first appear at second order. With particular boundary conditions, these second-order field equations may then restrict the allowed first-order perturbations; see [25, 26] for simple examples.

Maloney, Song, and Strominger have argued in [19] that such a phenomenon occurs for topologically massive gravity at the chiral coupling. Their analysis used the nonlocal light front modes of [18], making the computations rather complex. The present B-F scalar formalism allows a straightforward check of their results.

[‡]This choice does not eliminate the $\ln z$ asymptotics in (3.6), since the relevant term reappears as $\gamma^+$ in (2.8).

We begin with some simple manipulations of the exact field equations. Define

$$\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu}. \tag{4.1}$$

The full field equations of topologically massive gravity at the chiral coupling are then

$$E_{\mu\nu} = \mathcal{G}_{\mu\nu} + \frac{1}{2}\left(\epsilon_\mu{}^{\alpha\beta}\nabla_\alpha \mathcal{G}_{\beta\nu} + \epsilon_\nu{}^{\alpha\beta}\nabla_\alpha \mathcal{G}_{\beta\mu}\right) = 0. \tag{4.2}$$

Let $\xi^\mu$ be an arbitrary vector and define

$$\Delta_{\alpha\beta}[\xi] = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha, \qquad \chi^\rho[\xi] = -\xi^\rho - \frac{1}{2}\epsilon^{\rho\alpha\beta}\nabla_\alpha \xi_\beta. \tag{4.3}$$

$\Delta_{\alpha\beta}[\xi]$ measures the failure of $\xi^\mu$ to be a Killing vector; $\chi^\mu[\xi]$ measures the chirality of $\xi^\mu$, in the sense that in anti-de Sitter space,

$$\chi^\mu = \begin{cases} 0 & \text{if } \xi^\mu = \delta^\mu_+ \\ -\delta^\mu_z & \text{if } \xi^\mu = \delta^\mu_z \\ -2\delta^\mu_- & \text{if } \xi^\mu = \delta^\mu_- \end{cases}$$

As I show in the appendix, it is then an exact equality that

$$\xi^\mu E_\mu{}^\nu = -\chi^\mu \mathcal{G}_\mu{}^\nu + \frac{1}{2}\epsilon^{\nu\alpha\beta}\Delta_{\mu\beta}\mathcal{G}_\alpha{}^\mu + \nabla_\alpha \mathcal{F}^{\alpha\nu} \quad \text{with } \mathcal{F}^{\alpha\nu} = \epsilon^{\nu\alpha\beta}\xi_\mu\left(\mathcal{G}_\beta{}^\mu - \frac{1}{2}\delta^\mu_\beta \mathcal{G}\right). \tag{4.4}$$

This is of the right form for a linearization instability: if one expands around anti-de Sitter space, for which $\bar{\mathcal{G}}_\mu{}^\nu = 0$, the first-order "bulk" contribution to the smeared field equation

$$I_\xi = \int d^2x \sqrt{|g|}\, \xi^\mu E_\mu{}^t = 0 \tag{4.5}$$

vanishes if $\xi^\mu$ is an AdS Killing vector ($\bar{\Delta}_{\mu\nu} = 0$) of the proper chirality ($\bar{\chi}^\mu = 0$). Suppose in addition that boundary conditions force the contribution from $\mathcal{F}^{\alpha\nu}$ to vanish. Then, as above, a particular combination of the equations of motion is identically zero at first order, and new restrictions may be expected at second order.

To see whether such restrictions really occur, we must evaluate the second-order contribution $I^{(2)}_\xi$. Let us choose $\xi^\mu = \delta^\mu_+$. Then $\Delta_{\mu\nu}$, $\chi^\mu$, and $\mathcal{G}_\mu{}^\nu$ all vanish at zeroth order, so from (4.4)

$$\begin{aligned}
\sqrt{|g|}\,\xi^\mu E^{(2)+}_\mu &= -z^2(\sqrt{|g|}\,\chi^\mu)^{(1)}\mathcal{H}_{\mu-} + \frac{z^2}{2}\left(\Delta^{(1)}_{+-}\mathcal{H}_{z-} + \Delta^{(1)}_{--}\mathcal{H}_{z+}\right) + \partial_\alpha(z^{-3}\mathcal{F}^{(2)\alpha+}) \\
\sqrt{|g|}\,\xi^\mu E^{(2)-}_\mu &= -z^2(\sqrt{|g|}\,\chi^\mu)^{(1)}\mathcal{H}_{\mu+} - \frac{z^2}{2}\left(\Delta^{(1)}_{++}\mathcal{H}_{z-} + \Delta^{(1)}_{+-}\mathcal{H}_{z+}\right) + \partial_\alpha(z^{-3}\mathcal{F}^{(2)\alpha-}),
\end{aligned} \tag{4.6}$$

where $\mathcal{H}_{\mu\nu} = \mathcal{G}^{(1)}_{\mu\nu}$ is the first-order curvature. Moreover, from (4.3),

$$(\sqrt{|g|}\,\chi^\mu)^{(1)} = -\frac{1}{z}h_{+-}\xi^\mu - \frac{1}{2}\tilde{\epsilon}^{\mu\alpha\beta}\partial_\alpha h_{\beta+}, \quad \Delta^{(1)}_{\mu\nu} = \partial_+ h_{\mu\nu}. \tag{4.7}$$

Now, recall from section 3 that a general solution $h_{ij}$ of the linearized field equations consists of three pieces, one determined by an extremal B-F scalar, one formally equivalent to a diffeomorphism, and one a chiral pp-wave depending on $\ln z$. Of these, only the first permits arbitrary initial data of compact support, so let us begin with that piece. I show in the appendix that

$$\begin{aligned}\sqrt{|g|}\,\xi^\mu E^{(2)t}_\mu[X] = &-\frac{1}{\sqrt{2}}\frac{1}{z}\left[\left(\partial_z N(N+2)^2X\right)^2 + 2\left(\partial_+ N(N+2)^2X\right)^2\right] \\ &+ \partial_\alpha(z^{-3}\mathcal{F}^{(2)\alpha t}) + \textit{spatial boundary terms},\end{aligned} \tag{4.8}$$

where the boundary terms vanish if $X$ has compact support. The integral $I_\xi$ of (4.5) is thus

$$I^{(2)}_\xi = -\frac{1}{\sqrt{2}}\int dx\,dz\,\frac{1}{z}\left[\left(\partial_z N(N+2)^2X\right)^2 + 2\left(\partial_+ N(N+2)^2X\right)^2\right] + \int_{z=0} dx\, z^{-3}\mathcal{F}^{(2)zt}. \tag{4.9}$$

In general, one must also consider terms involving the remaining two pieces in $\tilde{h}_{ij}$, but I show in the appendix that these give only a boundary term that vanishes if $X$ has compact support.

In accord with the results of [19], we see that the "bulk" term in (4.9) is negative definite. The "boundary" term is precisely the Abbott-Deser-Tekin charge $Q^+$ [10, 27]. For the general logarithmic boundary conditions of [13, 23],

$$\begin{aligned} h_{++} &\sim A_{++}(x^+,x^-) + B(x^+)\ln z + \dots \\ h_{+-} &\sim A_{+-}(x^+,x^-) + \dots \\ h_{--} &\sim A_{--}(x^-) + \dots, \end{aligned} \tag{4.10}$$

it is easily checked that

$$Q^+ = -\frac{1}{\sqrt{2}}\int_{z=0} dx\, B. \tag{4.11}$$

By the equations of motion, the total integral $I_\xi$ must vanish. Thus if our perturbation expansion remains valid throughout the Poincaré patch (so that we can separately require that $I^{(2)}_\xi = 0$) and if we can consistently restrict ourselves to Brown-Henneaux boundary conditions $B(x^+) = 0$, the bulk integrand in (4.9) must be zero:

$$\partial_z N(N+2)^2X = \partial_+ N(N+2)^2X = 0. \tag{4.12}$$

The general solution of (2.5) and (4.12) is

$$X = \frac{\alpha_0}{z^2} + \frac{\beta_0\ln z}{z^2} + \frac{1}{2}\partial_+\partial_-(\beta_0 - \alpha_0) + \frac{1}{2}\partial_+\partial_-\beta_0\ln z \quad \text{with } {\partial_+}^2{\partial_-}^2\alpha_0 = {\partial_+}^2\partial_-\beta_0 = 0. \tag{4.13}$$

The resulting linearized curvature has nonvanishing components

$$\mathcal{H}_{--} = \frac{8{\partial_-}^2\beta_0}{z^2} + 2\partial_+\partial_-^3\beta_0, \qquad \mathcal{H}_{-z} = \frac{4\partial_+{\partial_-}^2\beta_0}{z}. \tag{4.14}$$

In particular, if we impose any reasonable AdS boundary conditions, we must require that ${\partial_-}^2\beta_0 = 0$, which in turn implies that the entire linearized curvature tensor is zero.

We thus confirm the results of [19]: if our perturbation expansion remains valid at $z = 0$, and if the sector $Q^+ = 0$ can be consistently treated as a superselection sector, then this sector exhibits a linearization instability, and only those linearized solutions with vanishing curvature

can be extended to second order. Equivalently, if a metric perturbation with nonvanishing $\mathcal{H}_{\mu\nu}$ has compact support at first order, the field equations force the second order perturbation to have a nonvanishing boundary contribution to the charge $Q^+$, and hence a nonvanishing logarithmic term in the asymptotic expansion (4.10). If such a logarithmic term can be consistently forbidden by boundary conditions, such first order metric perturbations are thus excluded.

Finally, let us note one more connection between chiral topologically massive gravity and scalar fields. The integral (4.9) that controls linearization instability depends on $X$ only through

$$\phi = \sqrt{2}N(N+2)^2X. \tag{4.15}$$

It is easily checked from (2.5) that $\phi$ obeys the equations of motion for a *massless* scalar field. Furthermore, the "bulk" term in (4.9) is simply the stress-energy tensor for $\phi$:

$$-\frac{1}{\sqrt{2}}\frac{1}{z}\left[\left(\partial_z N(N+2)^2X\right)^2 + 2\left(\partial_+ N(N+2)^2X\right)^2\right] = \sqrt{|g|}T_+{}^t[\phi]. \tag{4.16}$$

It was noted in [18] that topologically massive gravity in light front coordinates can be described in terms of a single massless scalar field, at the expense of allowing nonlocal dependence of $h_{\mu\nu}$ on the field. We now see that the same is true in the present setting.

## 5 Where we stand

The heart of this paper has been a demonstration that the linearized solutions of topologically massive AdS gravity at the chiral coupling may be parametrized by a free scalar field at the Breitenlohner-Freedman mass bound. The peculiarities of the chiral coupling—in particular, the appearance of logarithmic solutions—reflect the behavior of such an extremal scalar. While light front gauge provides an alternative scalar parametrization [18], the present form has the advantage of locality: the metric perturbations are now strictly local functions of the scalar $X$. Although this work has been carried out in a single Poincaré coordinate patch, there are no obvious obstructions to a similar construction in global coordinates. A scalar field at the B-F bound also appears in the description of exact pp-waves at the chiral coupling [28]; a further exploration of this relationship could be interesting.

The importance of this new parametrization, of course, depends on what it can tell us about the physical puzzles of chiral topologically massive gravity. As we have seen, we can write down the general solution of the linearized equations of motion, and use this to confirm the claim of [19] that the $Q^+ = 0$ sector has a linearization instability that excludes the propagating bulk modes. At the same time, the positive energy theorem of [9, 10] suggests that another superselection sector may exist, in which black holes are excluded. We are thus left with a picture of a theory containing three sectors: a chiral ($Q^+ = 0$) sector with black holes but no bulk modes, requiring $G > 0$ for positive energy; a sector with bulk modes but no black holes, requiring $G < 0$ for positive energy; and a larger sector with the logarithmic boundary conditions of [13,23], in which the energy may not be bounded below for any $G$.

To show this picture is correct, though, more work is needed. First, we do not know that the potential positive energy sectors are genuine superselection sectors that really decouple from the rest of the theory. In the chiral sector, the remaining linearized excitations are formal diffeomorphisms, which extend to exact solutions that are nontrivial only at the conformal boundary. Until the boundary dynamics is better understood, however, we cannot exclude the possibility that consistent interactions require that the $Q^+ \neq 0$ "bulk" modes be present at the

boundary as well. In the "no black holes" sector, it is plausible that the positive energy bulk excitations cannot collapse to form negative energy black holes, but I know no proof that such processes are excluded. In the "logarithmic" sector, we know that energies are unbounded below at low orders of perturbation theory, but not whether nonperturbative bounds exist.

Indeed, apart from investigations of the constraints and of solutions with special symmetries, work on this model has relied almost exclusively on perturbation theory. Efforts to prove, or disprove, a global positive energy theorem have not yet succeeded at the chiral coupling [29], and we cannot exclude the possibility of nonperturbative surprises. In particular, the low order analysis presented here, including the analysis of linearization instability, depends on the assumption that our perturbative expansion remains valid all the way out to $z = 0$, despite the presence of terms in the expansion with inverse powers of $z$. In some crude sense, we know this is not correct: if we choose a first-order solution with compact initial data, the second-order solution is generically nonzero all the way out to $z = 0$, so near the boundary $h^{(2)}$ dominates $h^{(1)}$. At this order, this phenomenon is merely another indication of linearization instability, and it is plausible that it does not extend to higher orders, but a more careful and rigorous analysis is clearly needed.

Finally, the failure of asymptotically anti-de Sitter space to be globally hyperbolic has some subtle implications that may not be fully appreciated. As in a globally hyperbolic spacetime, one may start with initial data in a compact region and evolve it forward in time with a Greens function of one's choice to obtain a solution of the field equations near the initial time slice. In a non-globally hyperbolic spacetime, however, the resulting solution may not depend continuously on the initial data; that is, arbitrarily small changes in initial data may lead to large changes in the solution, invalidating any perturbative expansion [30].[§] For a scalar field in anti-de Sitter space, Ishibashi and Wald have shown that this problem can be circumvented with a suitable choice of Greens functions [22], and I have used this result to determine the evolution of the scalar field $X$. But while the metric perturbations depend locally on $X$, the converse is not true, and it is not obvious that the evolution described here will always depend continuously on the initial data. For the full nonlinear theory, the situation is even less clear. Again, a much more careful and rigorous analysis is needed before we can be truly confident of any conclusions.

## Appendix. Details of some calculations

In this appendix, I describe my conventions and show some of the details of calculations described in the main body of the paper.

### a. Field equations

As in the main text, I work in a Poincaré coordinate patch, with $\Lambda = -1/\ell^2 = -1$. My metric signature is $-++$, with $\tilde{\epsilon}^{+-z} = \sqrt{|g|}\epsilon^{+-z} = -1$. As in [18], I use the "wrong sign" Newton's constant $G < 0$, which connects smoothly to the standard choice at $\Lambda = 0$. For solutions of the field equations, changing the sign of $G$ is equivalent to reversing chirality, so my "+" components are the "−" components of [7, 19].

In the Fefferman-Graham gauge (2.6), the linearization of the cosmological Einstein tensor

[§]See Ref. [31], chapter III §6 for a simple example of how this can occur for an elliptic differential equation.

$\mathcal{G}_{\mu\nu} = G_{\mu\nu} - g_{\mu\nu}$ around anti-de Sitter space is

$$\begin{aligned}
\mathcal{H}_{++} &= -\frac{1}{2}N(N+2)h_{++} & \mathcal{H}_{+z} &= \frac{z}{2}(N+2)(\partial_- h_{++} - \partial_+ h_{+-}) \\
\mathcal{H}_{+-} &= \frac{1}{2}N(N+2)h_{+-} & \mathcal{H}_{-z} &= \frac{z}{2}(N+2)(\partial_+ h_{--} - \partial_- h_{+-}) \\
\mathcal{H}_{--} &= -\frac{1}{2}N(N+2)h_{--} & \mathcal{H}_{zz} &= -(N+2)h_{+-} - \frac{z^2}{2}\left(\partial_-^2 h_{++} - 2\partial_+\partial_- h_{+-} + \partial_+^2 h_{--}\right).
\end{aligned} \tag{A.1}$$

The linearized field equations (4.2) are

$$\mathcal{H}_{\mu\nu} + \epsilon_\mu{}^{\alpha\beta}\bar{\nabla}_\alpha \mathcal{H}_{\beta\nu} = 0. \tag{A.2}$$

(The second term may be symmetrized in $\mu$ and $\nu$, but this need not be done explicitly, since the antisymmetric part vanishes by virtue of the Bianchi identities.) The six independent components of (A.2) may be taken to be

$$\begin{aligned}
E^{(1)}_{++} &= z\partial_+\mathcal{H}_{z+} - N\mathcal{H}_{++} = \frac{z^2}{2}(N+2)(\partial_+\partial_- h_{++} - \partial_+{}^2 h_{+-}) + \frac{1}{2}N^2(N+2)h_{++} \\
E^{(1)}_{+-} - \mathcal{E} &= -z\partial_-\mathcal{H}_{z+} + N\mathcal{H}_{+-} = -\frac{z^2}{2}(N+2)(\partial_-{}^2 h_{++} - \partial_+\partial_- h_{+-}) + \frac{1}{2}N^2(N+2)h_{+-} \\
E^{(1)}_{--} &= -z\partial_-\mathcal{H}_{z-} + (N+2)\mathcal{H}_{--} = -\frac{z^2}{2}(N+2)(\partial_+\partial_- h_{--} - \partial_-{}^2 h_{+-}) - \frac{1}{2}N(N+2)^2 h_{--} \\
E^{(1)}_{+z} &= z(\partial_-\mathcal{H}_{++} - \partial_+\mathcal{H}_{+-}) = -\frac{z}{2}N(N+2)(\partial_- h_{++} + \partial_+ h_{+-}) \\
E^{(1)}_{-z} + z\partial_-\mathcal{E} &= 2z\partial_-\mathcal{H}_{+-} + (N+1)\mathcal{H}_{-z} = \frac{z}{2}\left[(N+2)^2\partial_+ h_{--} + (N-2)(N+2)\partial_- h_{+-}\right] \\
\mathcal{E} &= 2\mathcal{H}_{+-} + \mathcal{H}_{zz} = (N-1)(N+2)h_{+-} - \frac{z^2}{2}(\partial_-{}^2 h_{++} - 2\partial_+\partial_- h_{+-} + \partial_+{}^2 h_{--}).
\end{aligned} \tag{A.3}$$

where $N = z\partial_z$ and $\mathcal{E} = \bar{g}^{\mu\nu}E^{(1)}_{\mu\nu}$.

## b. General solution

Our next goal is to find the general solution of (A.2). Without loss of generality, we may parametrize the metric perturbation $h_{+-}$ as

$$h_{+-} = 2(N+2)\partial_+\partial_- Z, \tag{A.4}$$

where $Z$ is an arbitrary function of $x^+$, $x^-$, and $z$. Then

$$\begin{aligned}
E^{(1)}_{-z} = 0 \quad &\Rightarrow (N+2)^2\partial_+\left[h_{--} + 2(N-2)\partial_-{}^2 Z\right] = 0 \\
&\Rightarrow h_{--} = -2(N-2)\partial_-{}^2 Z + A \quad \text{with } (N+2)^2\partial_+ A = 0, \\
E^{(1)}_{+z} = 0 \quad &\Rightarrow N(N+2)\partial_-\left[h_{++} + 2(N+2)\partial_+{}^2 Z\right] = 0 \\
&\Rightarrow h_{++} = -2(N+2)\partial_+{}^2 Z + B \quad \text{with } N(N+2)\partial_- B = 0.
\end{aligned} \tag{A.5}$$

The conditions on the functions $A$ and $B$ require that

$$\begin{aligned}
A &= \frac{\partial_-{}^2 a_1}{z^2} + \frac{\partial_-{}^2 a_2 \ln z}{z^2} + v^-(x^-, z) \\
B &= \frac{\partial_-{}^2 b_1}{z^2} + \partial_+{}^3\partial_- b_2 + v^+(x^+, z)
\end{aligned} \tag{A.6}$$

where $a_1$, $a_2$, $b_1$, and $b_2$ are arbitrary functions of $x^+$ and $x^-$; the derivatives $\partial_\pm$ are inserted for later notational convenience, and do not affect the generality of the solution. The equations $\mathcal{E}^{(1)} = 0$ and $E^{(1)}_{+-} = 0$ then become

$$(N-1)\partial_+\partial_- Y = \frac{z^2}{4}({\partial_-}^2 B + {\partial_+}^2 A), \qquad N^2\partial_+\partial_- Y = \frac{z^2}{2}(N+2){\partial_-}^2 B \tag{A.7}$$

with

$$Y = 2z^2\partial_+\partial_- Z + (N+2)^2 Z. \tag{A.8}$$

These are straightforward to integrate, yielding

$$Y = \frac{z^2}{4}{\partial_+}^2{\partial_-}^2 b_2 - \frac{1}{4}\partial_+\partial_-(a_1+a_2+b_1) + (N+2)^2 w^+(x^+,z) + (N+2)^2 w^-(x^-,z) \tag{A.9}$$

where the $w^\pm$ are arbitrary functions of their arguments. The remaining field equations $E^{(1)}_{++} = E^{(1)}_{--} = 0$ then reduce to

$$\begin{aligned} N^2(N+2)\left[v^+ - 2(N+2){\partial_+}^2 w^+\right] = 0 &\Rightarrow v^+ - 2(N+2){\partial_+}^2 w^+ = \frac{\alpha^+}{z^2} + \beta^+ + \gamma^+ \ln z \\ N(N+2)^2\left[v^- - 2(N-2){\partial_-}^2 w^-\right] = 0 &\Rightarrow v^- - 2(N-2){\partial_-}^2 w^- = \frac{\alpha^-}{z^2} + \beta^- + \frac{\gamma^- \ln z}{z^2} \end{aligned} \tag{A.10}$$

where $(\alpha^+, \beta^+, \gamma^+)$ and $(\alpha^-, \beta^-, \gamma^-)$ are arbitrary functions of $x^+$ and $x^-$, respectively.

Our next step is to solve (A.8) for $Z$, given the source (A.9). It is easy to check that a particular solution is

$$Z_0 = -\frac{a_1+a_2+b_1+2b_2}{8}\cdot\frac{1}{z^2} - \frac{a_2}{8}\cdot\frac{\ln z}{z^2} + \frac{1}{8}\partial_+\partial_- b_2 + w^+ + w^-. \tag{A.11}$$

The general solution will be of the form

$$Z = Z_0 + X, \tag{A.12}$$

where $X$ is a solution of the homogeneous equation (2.5).

Finally, we insert (A.6), (A.10), (A.11), and (A.12) into (A.4)–(A.6) to determine the full first-order metric perturbation. A straightforward computation now yields the result (2.7)–(2.8), where $X$ and $\gamma^+$ in (2.7) and (2.8) are as in (A.12) and (A.10), and $a$, $b$, and $c$ in (2.8) are linear combinations of $a_1$, $a_2$, $b_1$, and $b_2$ in (A.6).

## c. Linearization instability

I next turn to the derivation of equation (4.4). I will make liberal use of the identity, true for any $B_{\mu\nu}$, that

$$B_{\mu\nu} - B_{\nu\mu} = -\epsilon_{\mu\nu\rho}\epsilon^{\rho\sigma\tau}B_{\sigma\tau}$$

(where the sign on the right-hand side follows from my "mostly minus" metric signature convention). We then have

$$\begin{aligned}
\xi^\mu E_\mu{}^\nu =& -\chi^\mu \mathcal{G}_\mu{}^\nu - \frac{1}{2}\epsilon^{\mu\alpha\beta}\nabla_\alpha \xi_\beta \mathcal{G}_\mu{}^\nu + \frac{1}{2}\epsilon^{\mu\alpha\beta}\xi_\mu \nabla_\alpha \mathcal{G}_\beta{}^\nu + \frac{1}{2}\epsilon^{\nu\alpha\beta}\xi_\mu \nabla_\alpha \mathcal{G}_\beta{}^\mu \\
=& -\chi^\mu \mathcal{G}_\mu{}^\nu - \frac{1}{2}\nabla_\alpha\left(\epsilon^{\mu\alpha\beta}\xi_\beta \mathcal{G}_\mu{}^\nu - \epsilon^{\mu\nu\beta}\xi_\beta \mathcal{G}_\mu{}^\alpha\right) \\
&- \frac{1}{2}\epsilon^{\mu\nu\beta}\nabla_\alpha \xi_\beta \mathcal{G}_\mu{}^\alpha + \frac{1}{2}\epsilon^{\mu\alpha\beta}\left(\xi_\beta \nabla_\alpha \mathcal{G}_\mu{}^\nu + \xi_\mu \nabla_\alpha \mathcal{G}_\beta{}^\nu\right) + \frac{1}{2}\epsilon^{\nu\alpha\beta}\xi_\mu \nabla_\alpha \mathcal{G}_\beta{}^\mu \qquad (A.13) \\
=& -\chi^\mu \mathcal{G}_\mu{}^\nu - \frac{1}{2}\nabla_\alpha\left(\epsilon^{\mu\alpha\beta}\xi_\beta \mathcal{G}_\mu{}^\nu - \epsilon^{\mu\nu\beta}\xi_\beta \mathcal{G}_\mu{}^\alpha - \epsilon^{\nu\alpha\beta}\xi_\mu \mathcal{G}_\beta{}^\mu\right) \\
&- \frac{1}{2}\left(\epsilon^{\alpha\nu\beta}\nabla_\mu \xi_\beta \mathcal{G}_\alpha{}^\mu + \epsilon^{\nu\alpha\beta}\nabla_\alpha \xi_\mu \mathcal{G}_\beta{}^\mu\right) = -\chi^\mu \mathcal{G}_\mu{}^\nu + \frac{1}{2}\epsilon^{\nu\alpha\beta}\Delta_{\mu\beta}\mathcal{G}_\alpha{}^\mu + \nabla_\alpha \mathcal{F}^{\alpha\nu}
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{F}^{\alpha\nu} &= -\frac{1}{2}\left(\epsilon^{\mu\alpha\beta}\xi_\beta \mathcal{G}_\mu{}^\nu - \epsilon^{\mu\nu\beta}\xi_\beta \mathcal{G}_\mu{}^\alpha - \epsilon^{\nu\alpha\beta}\xi_\mu \mathcal{G}_\beta{}^\mu\right) = \frac{1}{2}\epsilon^{\alpha\nu\sigma}\epsilon_{\sigma\lambda\tau}\epsilon^{\mu\lambda\beta}\xi_\beta \mathcal{G}_\mu{}^\tau + \frac{1}{2}\epsilon^{\nu\alpha\beta}\xi_\mu \mathcal{G}_\beta{}^\mu \\
&= -\frac{1}{2}\epsilon^{\alpha\nu\sigma}\delta^{\mu\beta}_{\sigma\tau}\xi_\beta \mathcal{G}_\mu{}^\tau + \frac{1}{2}\epsilon^{\nu\alpha\beta}\xi_\mu \mathcal{G}_\beta{}^\mu = \epsilon^{\nu\alpha\beta}\xi_\nu\left(\mathcal{G}_\beta{}^\mu - \frac{1}{2}\delta^\mu_\beta \mathcal{G}\right). \qquad (A.14)
\end{aligned}$$

I next turn to equation (4.9). Note first that from (4.7)

$$\begin{aligned}
(\sqrt{|g|}\,\chi^+)^{(1)} &= -\frac{1}{2}\frac{1}{z}(N+2)h_{+-} \\
(\sqrt{|g|}\,\chi^-)^{(1)} &= \frac{1}{2}\frac{1}{z}N\,h_{++} \qquad (A.15) \\
(\sqrt{|g|}\,\chi^-)^{(1)} &= \frac{1}{2}(\partial_+ h_{+-} - \partial_- h_{++}).
\end{aligned}$$

Let us start by considering the quantity $\sqrt{|g|}\,\xi^\mu E_\mu^{(2)-}$. From (4.7) and (A.1), the terms on the right-hand side of (4.6) are

$$\begin{aligned}
-z^2(\sqrt{|g|}\,\chi^\mu)^{(1)}\mathcal{H}_{\mu+} =& -\frac{z}{4}(N+2)h_{+-}\cdot(N+2)Nh_{++} - \frac{z}{4}N(N+2)h_{+-}\cdot Nh_{++} \\
&+ \frac{z^3}{4}(\partial_+ h_{+-} - \partial_- h_{++})(N+2)(\partial_+ h_{+-} - \partial_- h_{++}) \qquad (A.16) \\
=& -\partial_z\left[\frac{z^2}{4}(N+2)h_{+-}\cdot Nh_{++} - \frac{z^4}{8}\left(\partial_+ h_{+-} - \partial_- h_{++}\right)^2\right] \\
-\frac{z^2}{2}\Delta^{(1)}_{+-}\mathcal{H}_{z+} =& -\frac{z^3}{4}\partial_+ h_{+-}(N+2)(\partial_- h_{++} - \partial_+ h_{+-}) \qquad (A.17) \\
=&\ \partial_z\left[\frac{z^4}{4}\left(\partial_+ h_{+-}\right)^2\right] - \frac{z^3}{4}\partial_+ h_{+-}(N+2)(\partial_- h_{++} + \partial_+ h_{+-}) \\
-\frac{z^2}{2}\Delta^{(1)}_{++}\mathcal{H}_{z-} =& -\frac{z^2}{2}\partial_+(h_{++}\mathcal{H}_{z-}) + \frac{z}{4}h_{++}(N-2)N(N+2)h_{+-} \\
=& -\frac{z^2}{2}\partial_+(h_{++}\mathcal{H}_{z-}) - \frac{z}{4}h_{+-}N(N+2)(N+4)h_{++} \qquad (A.18) \\
&+ \partial_z\left[\frac{z^2}{4}\left(h_{++}(N-2)Nh_{+-} - Nh_{++}(N-2)h_{+-} + h_{+-}N(N+2)h_{++}\right)\right]
\end{aligned}$$

where I have used the equation of motion $\partial_+ \mathcal{H}_{z-} = \frac{1}{z}(N-2)\mathcal{H}_{+-}$ in the first line. Now let

$$\begin{aligned} V =& -\frac{z^2}{2} h_{++}\mathcal{H}_{z-} - \frac{z^3}{2} h_{+-}(N+6)\partial_+ h_{+-} \\ \partial_+ V =& -\frac{z^2}{2}\partial_+(h_{++}\mathcal{H}_{z-}) - \frac{z^3}{2}\partial_+ h_{+-}(N+6)\partial_+ h_{+-} \\ & - \frac{z^3}{2}(N+6)\left(\partial_+{}^2 h_{+-} - \frac{1}{2z^2}N(N+2)h_{++}\right) h_{+-} - \frac{z}{4} h_{+-}N(N+2)(N+4)h_{++} \\ =& -\frac{z^2}{2}\partial_+(h_{++}\mathcal{H}_{z-}) - \partial_z\left[\frac{z^4}{4}\left(\partial_+ h_{+-}\right)^2\right] - z^3\left(\partial_+ h_{+-}\right)^2 \\ & - \frac{z^3}{2} h_{+-}(N+6)\left(\partial_+{}^2 h_{+-} - \frac{1}{2z^2}N(N+2)h_{++}\right) - \frac{z}{4} h_{+-}N(N+2)(N+4)h_{++}. \end{aligned} \tag{A.19}$$

Inserting (A.16)–(A.19) into (4.6), we see that

$$\begin{aligned} \sqrt{|g|}\,\xi^\mu E_\mu^{(2,\mathrm{bulk})-} =& \; z^3\left(\partial_+ h_{+-}\right)^2 + \partial_+ V + \partial_z U^- \\ & + \frac{z^3}{2} h_{+-}(N+6)\left(\partial_+{}^2 h_{+-} - \frac{1}{2z^2}N(N+2)h_{++}\right) - \frac{z^3}{4}\partial_+ h_{+-}(N+2)(\partial_- h_{++} + \partial_+ h_{+-}), \end{aligned} \tag{A.20}$$

where $V$ is as in (A.19) and

$$\begin{aligned} U^- =& \; \frac{z^4}{2}(\partial_+ h_{+-})^2 + \frac{z^4}{8}(\partial_+ h_{+-} - \partial_- h_{++})^2 \\ & + \frac{z^2}{4}\left(h_{++}(N-2)N h_{+-} - 2N h_{++} N h_{+-} + h_{+-}N(N+2)h_{++}\right). \end{aligned} \tag{A.21}$$

We next turn to $\sqrt{|g|}\,\xi^\mu E_\mu^{(2)+}$, which by (4.6) and (4.7) is

$$\sqrt{|g|}\,\xi^\mu E_\mu^{(2,\mathrm{bulk})+} = \frac{z}{2}(N+2)h_{+-}\mathcal{H}_{+-} - \frac{z}{2}N h_{++}\mathcal{H}_{--} + \frac{z^2}{2}\partial_- h_{++}\mathcal{H}_{z-} + \frac{z^2}{2}\partial_+ h_{--}\mathcal{H}_{z+}. \tag{A.22}$$

It is then straightforward to see from (A.19) that

$$\begin{aligned} \sqrt{|g|}\,\xi^\mu E_\mu^{(2,\mathrm{bulk})+} =& -\partial_- V + \frac{z}{4}(N+2)h_{+-}N(N+2)h_{+-} - \frac{z^3}{2}h_{+-}(N+6)\partial_+\partial_- h_{+-} \\ & - \frac{z^3}{2}\partial_- h_{+-}(N+6)\partial_+ h_{+-} + \frac{z^3}{4}\partial_+ h_{--}(N+2)(\partial_- h_{++} - \partial_+ h_{+-}) - \partial_z\left[\frac{z^2}{2}h_{++}\mathcal{H}_{--}\right] \\ =& \; -\partial_- V + z h_{+-}N(N+2)h_{+-} + \frac{z^3}{2}(N-2)h_{+-}\left[\partial_+\partial_- h_{+-} + \frac{1}{2z^2}N(N+2)h_{+-}\right] \\ & + \frac{z^3}{2}\left[(N-2)\partial_- h_{+-} + (N+2)\partial_+ h_{--}\right]\partial_+ h_{+-} + \frac{z^3}{4}\partial_+ h_{--}(N+2)(\partial_- h_{++} + \partial_+ h_{+-}) \\ & - \partial_z\left[\frac{z^2}{2}h_{++}\mathcal{H}_{--} + \frac{z^4}{4}\left(h_{+-}\partial_+\partial_- h_{+-} + \partial_+ h_{+-}\partial_- h_{+-} + \partial_+ h_{--}\partial_+ h_{+-}\right)\right] \\ =& \; -\partial_- V - \partial_z U^+ - z\left[(N+2)h_{+-}\right]^2 + \frac{z^3}{2}(N-2)h_{+-}\left[\partial_+\partial_- h_{+-} + \frac{1}{2z^2}N(N+2)h_{+-}\right] \\ & + \frac{z^3}{2}\left[(N-2)\partial_- h_{+-} + (N+2)\partial_+ h_{--}\right]\partial_+ h_{+-} + \frac{z^3}{4}\partial_+ h_{--}(N+2)(\partial_- h_{++} + \partial_+ h_{+-}) \end{aligned} \tag{A.23}$$

with

$$U^+ = \frac{z^2}{2} h_{++}\mathcal{H}_{--} - z^2 h_{+-}(N+2)h_{+-} + \frac{z^4}{4}\left(h_{+-}\partial_+\partial_- h_{+-} + \partial_+ h_{+-}\partial_- h_{+-} + \partial_+ h_{--}\partial_+ h_{+-}\right). \tag{A.24}$$

Now recall first that at first order, $h$ has two contributions (2.7) and (2.8). The integral $I_\xi^{(2)}$ will correspondingly have three contributions: one quadratic in $X$, one quadratic in $\tilde{h}$, and one cross term. Let us first consider the term quadratic in $X$. From (2.7),

$$\begin{aligned}
\partial_+{}^2 h_{+-} - \frac{1}{2z^2}N(N+2)h_{++} &= 0 \\
\partial_- h_{++} + \partial_+ h_{+-} &= 0 \\
\partial_+\partial_- h_{+-} + \frac{1}{2z^2}N(N+2)h_{+-} &= 0 \\
(N-2)\partial_- h_{+-} + (N+2)\partial_+ h_{--} &= 0.
\end{aligned} \tag{A.25}$$

Inserting (A.25) into (A.20) and (A.23) and combining to form the $t$ component $\sqrt{|g|}\,\xi^\mu E_\mu^{(2,\mathrm{bulk})t}$, we immediately obtain (4.9).

We next consider the contributions involving $\tilde{h}$. We could again use (A.20) and (A.23), but it is simpler to return to (4.6). Observe that the only nonvanishing contribution of $\tilde{h}$ to the linearized curvature is $\widetilde{\mathcal{H}}_{++} = -\gamma^+$, so (4.6) will consist almost entirely of cross terms involving $\mathcal{H}[X]$. Note also that if $f(x^+, x^-)$ is any $z$-independent function, then it follows from (A.1) that $zf\mathcal{H}_{++}$, $zf\mathcal{H}_{+-}$, $zf\mathcal{H}_{--}$, $z^{-1}f\mathcal{H}_{+-}$, $z^{-1}f\mathcal{H}_{--}$, $f\mathcal{H}_{z+}$, and $f\mathcal{H}_{z-}$ are all total $z$ derivatives. It is then easy to check that the only remaining $\tilde{h}$-dependent contributions to $\sqrt{|g|}\,\xi^\mu E_\mu^{(2)+}$ are

$$\sqrt{|g|}\,\xi^\mu E_\mu^{(2,\mathrm{bulk})+} = \cdots + \frac{z^2}{8}\left[\partial_+{}^3\partial_-{}^2(2a-b-c)\mathcal{H}_{z-} + \partial_+{}^2\partial_-{}^3(2a-b-c)\mathcal{H}_{z+}\right]. \tag{A.26}$$

But

$$\begin{aligned}
\mathcal{H}_{z-} &= \frac{z}{2}(N+2)(\partial_+ h_{--} - \partial_- h_{+-}) = -2zN(N+2)\partial_+\partial_-{}^2 X = \frac{1}{z}(N-2)N(N+2)^2\partial_- X \\
\mathcal{H}_{z+} &= \frac{z}{2}(N+2)(\partial_- h_{++} - \partial_+ h_{+-}) = -2z(N+2)^2\partial_+{}^2\partial_- X = \frac{1}{z}N^2(N+2)^2\partial_+ X,
\end{aligned} \tag{A.27}$$

so the terms in (A.26) are each of the form

$$zf(x^+, x^-)(N+2)g = \partial_z(z^2 fg).$$

Similarly, the only $\tilde{h}$-dependent contributions to $\sqrt{|g|}\,\xi^\mu E_\mu^{(2)-}$ that are not immediately recognizable as total $z$ derivatives are

$$\begin{aligned}
\sqrt{|g|}\,\xi^\mu E_\mu^{(2,\mathrm{bulk})-} = \ldots &- \frac{z^2}{8}\left[\partial_+{}^4\partial_-(2a-b-c)\mathcal{H}_{z-} + \partial_+{}^3\partial_-{}^2(2a-b-c)\mathcal{H}_{z+}\right] \\
&- \frac{z^2\ln z}{2}\partial_+\gamma^+\mathcal{H}_{z-} - \frac{z}{2}\gamma^+(N+2)h_{+-}.
\end{aligned} \tag{A.28}$$

As in (A.26), the first two terms are again total $z$ derivatives, as is the last term. The remaining term is also a total derivative, although less obviously:

$$\begin{aligned}
-\frac{z^2 \ln z}{2}\partial_+\gamma^+\mathcal{H}_{z-} &= -\frac{z \ln z}{2}\partial_+\gamma^+(N-2)N(N+2)^2\partial_- X \\
&= -\partial_z\left[\frac{z^2 \ln z}{2}\partial_+\gamma^+(N-2)N(N+2)\partial_- X\right] + \frac{z}{2}\partial_+\gamma^+(N-2)N(N+2)\partial_- X \\
&= -\partial_z\left[\frac{z^2 \ln z}{2}\partial_+\gamma^+(N-2)N(N+2)\partial_- X - \frac{z^2}{2}\partial_+\gamma^+(N-2)N\partial_- X\right].
\end{aligned} \tag{A.29}$$

We thus conclude that as long as $X$ has compact support, the terms involving $\tilde{h}$ make no contribution to the integral $I^{(2)}_\xi$, and the expression (4.9) is fully general.

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